Nonlinear dynamics of continuously tunable optoelectronic oscillators based on stimulated Brillouin amplification

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Abstract: We present a theoretical analysis for tunable optoelectronic oscillators (OEOs) based on stimulated Brillouin scattering (SBS). A pump laser is used to generate a Brillouin gain which selectively amplifies a phase-modulated and contra-propagating laser signal. The radiofrequency beatnote generated after photodetection is filtered, amplified and fed back to the phase modulator to close the optoelectronic loop. Tunability is readily achieved by the adjustable detuning of the pump and signal lasers. OEOs based on stimulated Brillouin scattering have been successfully demonstrated at the experimental level, and they feature competitive phase noise performances along with continuous tunability for the output radiofrequency signal, up to the millimeter-wave band. However, the nonlinear dynamics of SBS-based OEOs remains largely unexplored at this date. In this article, we propose a model that describes the temporal dynamics of the microwave envelope, thereby allowing us to track the dynamics of the amplitude and phase of the radiofrequency signal. The corresponding nonlinear and time-delayed differential equation is then analyzed to reveal the underlying bifurcation behavior that emerges as the feedback gain is increased. It is shown that after the primary Hopf bifurcation that triggers the microwave oscillations, the system undergoes a secondary Neimark-Sacker bifurcation before fully developed chaos emerges for the highest gain values. We also propose a model for the chipscale version of this SBS-based OEO where the delay line is replaced by a highly nonlinear waveguide. The numerical simulations are found to be in excellent agreement with the analytical study.

1. Introduction

Optoelectronic oscillators (OEOs) are microwave photonic systems that have found a plethora of applications in various disciplines related to microwave photonics (see review article [1]). The most important and most investigated application so far is the generation of ultra-pure radiofrequency (RF) signals [2–4]. In their most basic configuration, they feature a closed loop where an amplitude-modulated laser signal is launched into a few km-long optical fiber delay line before being photodetected, and the output microwave signal is frequency-filtered and amplified before being fed back to the light modulator [5]. This original architecture had the great advantage of simplicity and offers a competitive phase noise performance. However, since the filtering occurs in the RF domain, the resulting oscillator is generally not tunable in frequency. Another drawback is narrowband RF filtering in the mm-wave range (frequency beyond 30 GHz) becomes progressively less effective as the frequency increases.

　　Both these problems are addressed by OEOs based on stimulated Brillouin scattering (SBS). Using Brillouin scattering to improve the functionalities of OEOs is a widespread strategy that has been implemented in various platforms [6–9]. In the configuration of interest in this article, a pump laser is launched in an highly nonlinear fiber spool in order to generate a Brillouin gain profile at an adjacent redshifted frequency. This process permits the selective amplification of a phase-modulated and contra-propagating laser signal. Most importantly, the RF oscillation

frequency solely depends on the spectral gap between the pump and signal lasers, and is thereby easily tunable. From that regard, a main characteristic of this approach is that frequency selection is achieved by the means of a microwave photonic filter instead of an RF filter [10–12]. Once the RF beatnote is generated after photodetection, it is filtered, amplified and fed back to the phase modulator to close the optoelectronic loop. It has been demonstrated by several research groups that these SBS-based OEOs feature frequency tunability from DC to more than 60 GHz and an ultra-low phase noise figure [13–15]. In fact, the km-long fiber spool can be replaced by a sub-cm-long highly nonlinear waveguide, and in that case, the system has the potential to be reduced to a chipscale size [16].

Fig. 1. Schematic representation of the SBS-based OEO, for the generic case where $T \neq 0$. Circ: Circulator; Filter: RF filter intended to select the beatnote frequency Ω_{rf} (wideband); ISO: Isolator; HNLF: Highly nonlinear fiber; PC: Polarization-controller; PD: Photodetector; PM: Phase modulator; PS: Phase shifter; RFA: Radiofrequency amplifier.

Despite the potential and many distinctive advantages of SBS-based OEOs, no dynamical model has ever been proposed to describe their nonlinear behavior. Indeed, previous research has shown that it is in general possible to obtain an equation for the envelope of the microwave generated by these OEOs [1,17–19]. In this article, we undertake a theoretical analysis showing that an delay-differential envelope equation for SBS-OEOs can be derived, analogously to what had been done earlier for the conventional architectures of OEOs. In agreement is shown that as the gain is increased, the system undergoes a primary Hopf bifurcation and then a secondary Neimark-Sacker bifurcation before being driven to fully developed chaos.

The outline of the article paper is following. In Sec. 2, the SBS-based OEO is presented and its principle of operation is described. Section 3 is devoted to the analysis of the stimulated Brillouin scattering dynamics, and an input-output relationship is derived for the selectively amplified signal wave. The dynamical characterization of the SBS-based microwave photonic filter enables us to obtain a time-domain differential equation for the OEO in Sec. 4. Sections 5 and 6 are focused on evaluating the stability of the stationary dynamical states predicted by the model in the cases where the frequency-selective amplification is mediated by a km-long fiber delay line ($T \neq 0$) or by a sub-cm-long waveguide ($T \simeq 0$), respectively. The last section concludes the article.

2. System and operation principle

The schematic representation of the SBS-based OEO is presented in Fig. 1. The dynamics of this system can be described by two key variables. The first one relates to electric field $E(t)$ at the

input of the phase modulator, while the second relates to the voltage $V(t)$ feeding the modulator. Instead of working directly with the real-valued variables $E(t)$ and $V(t)$, we use instead their complex slowly-varying amplitudes $\mathcal{E}(t) = E(t)e^{i\varphi(t)}$ and $\mathcal{V}(t) = V(t)e^{i\psi(t)}$ defined through

$$E(t) = \frac{1}{2}\mathcal{E}(t)e^{i\omega_0 t} + \frac{1}{2}\mathcal{E}^*(t)e^{-i\omega_0 t} \tag{1}$$

$$V(t) = \frac{1}{2}\mathcal{V}(t)e^{i\Omega_0 t} + \frac{1}{2}\mathcal{V}^*(t)e^{-i\Omega_0 t}, \tag{2}$$

where $\omega_0 \equiv \omega_s$ and $\Omega_0 \equiv \Omega_{rf}$ are the angular frequencies associated with the signal infrared laser beam (around 1550 nm) and to the microwave signal (multi-GHz frequency to be defined later), respectively.

Fig. 2. Schematic Representation of a chipscale SBS-based OEO. This system corresponds to the case $T = 0$ in our analysis (see Sec. 4.4). Only the essential elements have been included in the figure. Circ: Circulator; HNLW: Highly nonlinear waveguide; PD: Photodetector. PM: Phase modulator; A proof-of-concept experiment for chispscale SBS-based OEOs has been presented in Ref. [16], using a chalcogenide waveguide.

The pump optical path is also called high-order frequency selection branch, and it is where the RF frequency Ω_0 is determined. The *pump* field is $E_p(t)$ while $E_s(t)$ is the phase-modulated *signal* field that is launched at the input of the highly nonlinear fiber. The two waves counter-propagate in the nonlinear fiber spool, and the pump wave stimulates Brillouin backscattering. The frequencies of the fields involved in this process are related as $\omega_p = \omega_s + \Omega_{rf} + \Omega_B$, so that the RF output frequency can be obtained as

$$\Omega_{rf} \equiv \Omega_0 = \omega_p - \omega_s - \Omega_B \tag{3}$$

where Ω_B is the Brillouin shift in the fiber (typically $2\pi \times 10$–20 GHz). In the oscillator, the first blue-sidemode is chosen for the SBS amplification. The peak of the Brillouin gain is such that $\omega_B = \omega_p - \Omega_B$, and the microwave photonic filter amplifies this mode while narrowing its linewidth [see Fig. 1(a)]. In Eq. (3), the frequencies ω_s and Ω_B are fixed: however, the tunability of ω_p allows the RF output frequency to be tunable as well. As a result of the frequency-selective amplification, the zeroth (central) and first-order (blue-side) modes of the phase-modulated signal beam become dominant [see Fig. 1(b)], and their spectral spacing defines the RF beatnote frequency after photodetction.

It is important to note that one of the most outstanding challenge in OEO technology is the generation of mm-waves [20–23]. In general, achieving such high frequencies (>30 GHz) is achieved via frequency multiplication [24–30]. Additionally, The other outstanding challenge for OEOs is tunability, which has been addressed using several different techniques [31–40]. From that perspective, a key advantage of SBS-based OEOs is that they addresses both challenges simultaneously and the only limitation for the maximal frequency of the RF signal is the bandwidth of the photodiode and phase modulator.

3. Frequency-selective amplification based on stimulated Brillouin scattering

3.1. Brillouin amplification

The phenomenology of frequency-selective Brillouin amplification is the following. We consider that the highly nonlinear fiber of length L and aligned along an axis z such that the fiber is located at $0 \leq z \leq L$. The pump laser wave

$$E_p(t) = \frac{1}{2}\mathcal{E}_p e^{i\omega_p t} + \frac{1}{2}\mathcal{E}_p^* e^{-i\omega_p t} \tag{4}$$

is launched on the right end (at $z = L$), with power $P_p = |\mathcal{E}_p|^2$ and optical frequency $\omega_p \simeq 2\pi \times 193$ THz (i. e. $\lambda_p \simeq 1550$ nm). As it propagates in the $-z$ direction, it creates a Lorentzian Brillouin gain profile

$$g_B(\omega) = g_{B0} \frac{(\Delta\Omega_B/2)^2}{(\Delta\Omega_B/2)^2 + (\omega - \omega_B)^2} \tag{5}$$

around the red-shifted frequency $\omega_B = \omega_p - \Omega_B$, where Ω_B is the Brillouin shift in the fiber (typically $2\pi \times 10$–20 GHz), $\Delta\Omega_B$ being the full width at half-maximum (FWHM) of the gain profile (typically $2\pi \times 10$–50 MHz) and g_{B0} being the peak value of the Brillouin gain (in m/W).

Consider now a contra-propagating signal laser beam $E_s(t)$ defined as

$$E_s(t) = \frac{1}{2}\mathcal{E}_s e^{i\omega_s t} + \frac{1}{2}\mathcal{E}_s^* e^{-i\omega_s t} , \tag{6}$$

that is launched on the left end (at $z = 0$), with power $P_s = |\mathcal{E}_s|^2$ and a frequency that falls within the bandwidth of the Brillouin gain, that is $\omega_s \simeq \omega_B$ [note that this simplified situation does *not* correspond to the one described in Fig. 1(a), where it is a sidemode of the phase-modulated signal that is amplified, so that we have instead $\omega_s \simeq \omega_B - \Omega_{rf}$; equivalently, one can say that both configurations are equivalent is $\Omega_{rf} \equiv 0$]. The signal wave would undergo frequency-selective amplification, i.e., the spectral components of the signal falling within the bandwidth will be amplified, while those that are outside would remain unchanged (provided we ignore all other effects in the fiber, such as losses, cross- and self-phase modulation, etc.). Since the fiber has a length L, it takes a time $T = L/v_g$ for the signal E_s to exit the fiber amplifier, with v_g being the group velocity of light in the fiber. The output signal would therefore a *selectively amplified* and *time-delayed* counterpart of the input signal, and the purpose of this section is to find the dynamical relationship between both.

3.2. Equations of the traveling waves

The time-domain slowly-varying envelope of the input and output signals are written as $\mathcal{E}_s^{in}(t)$ and $\mathcal{E}_s^{out}(t)$, respectively (in units of $\sqrt{W}$). We can introduce their corresponding intensities as $I_s^{in,out}(t) = |\mathcal{E}_s^{in,out}(t)|^2/A_{eff}$ (in units of W/m^2), with A_{eff} being the effective area of the fiber.

If we neglect dispersion and cross/self-phase modulation in the fiber, the pump and signal intensities $I_p(z,t)$ and $I_s(z,t)$ at any point of the Brillouin amplifier obey the well-known equations [41,42]

$$\frac{\partial I_p}{\partial z} - \frac{1}{v_g}\frac{\partial I_p}{\partial t} = g_{B0}I_s I_p + \alpha I_p \tag{7}$$

$$\frac{\partial I_s}{\partial z} + \frac{1}{v_g}\frac{\partial I_s}{\partial t} = g_{B0}I_p I_s - \alpha I_s , \tag{8}$$

where α stands for the linear losses in the fiber (in units of m^{-1}). Note unlike in Ref. [42], the z-coordinate is here oriented along the propagation direction of the *signal* laser wave (instead of the pump wave), for being the variable of interest in our study.

We furthermore make the following two assumptions: *(i)* The pump laser power is assumed undepleted (i. e. I_p is assumed approximately constant, and such that $I_p \gg I_s$); *(ii)* and the losses in the fiber are negligible in comparison to the Brillouin amplification (i. e., $\alpha \simeq 0$). As a consequence, the above equations are reduced to the single equation

$$\frac{\partial I_s}{\partial z} + \frac{1}{v_g}\frac{\partial I_s}{\partial t} = g_{B0}I_p I_s \quad \text{with} \quad I_p = \text{Const.}, \tag{9}$$

which has a generic solution of the form

$$I_s(z, t) = e^{g_{B0}I_p z} I\left(t - \frac{z}{v_g}\right) \tag{10}$$

where I is at this point an arbitrary waveform propagating along the fiber in the z direction. Indeed, the input and output signals in the fiber can be defined as

$$I_s^{in}(t) \equiv I_s(0, t) = I(t) \tag{11}$$

$$I_s^{out}(t) \equiv I_s(L, t) = e^{g_{B0}I_p L} I(t - T), \tag{12}$$

where $L = v_g T$ is the length of the Brillouin-active fiber yielding a time-delay T when light travels at the group velocity v_g. We therefore deduce

$$\frac{I_s^{out}(t)}{I_s^{in}(t - T)} = e^{g_{B0}I_p L} \tag{13}$$

and since $e^{g_{B0}I_p L} > 1$, it results as expected that owing to stimulated Brillouin scattering, the output signal is an amplified version of the delayed input signal.

3.3. Accounting for frequency selection

Equation (13) does not account for frequency selection in the amplification process; more precisely, it assumes that the pump and the input/output signals are perfectly monochromatic signals with center frequencies ω_p and $\omega_s = \omega_B$, respectively.

Accounting for the effect of the Brillouin gain *bandwidth* on the amplification process requires rewrite Eq. (13) in the Fourier domain and to replace the gain parameter g_{B0} by the Brillouin gain profile $g_B(\Omega)$, following

$$\frac{\tilde{I}_s^{out}(\Omega)}{\tilde{I}_s^{in}(\Omega)} = \frac{|\tilde{\mathcal{E}}_s^{out}(\Omega)|^2}{|\tilde{\mathcal{E}}_s^{in}(\Omega)e^{-i\Omega T}|^2} = e^{g_B(\Omega)I_p L}, \tag{14}$$

where $\Omega = \omega - \omega_B$ is the frequency detuning from the center frequency of the Brillouin gain. It is important to note here that according to Eq. (14), the Brillouin gain viewed by the incoming signal $\tilde{I}_s^{in}(\omega)$ is *not* a Lorentzian; Instead, it is the *exponential* of a Lorentzian. This gain profile, which can prosaically be described as a hump on top of a flat background equal to 1, provides a sound intuitive description of the Brillouin selective amplification. On the one hand, frequency components ω far outside the gain profile are not amplified [$g_B(\omega) \simeq 0$, i.e. $e^{g_B(\omega)I_p L} \simeq 1$] and the input signals are unaltered; On the other hand, frequency components ω inside the gain profile amplified [$g_B(\omega \simeq \omega_B) > 0$, i.e. $e^{g_B(\omega)I_p L} > 1$].

We need now to rewrite Eq. (14) in terms of fields in the Fourier domain, so that the phase information of all the signals involved can be retrieved. We first express the Brillouin gain as

$$g_B(\Omega) = g_{B0} |\mathcal{L}_B(\Omega)|^2, \tag{15}$$

where

$$\mathcal{L}_{\mathrm{B}}(\Omega) = \frac{(\Delta\Omega_{\mathrm{B}}/2)}{(\Delta\Omega_{\mathrm{B}}/2) + i\Omega} \tag{16}$$

is the complex-valued Lorentzian profile as viewed by the signal field. Using the equality

$$e^{g_{\mathrm{B}}(\Omega)I_{\mathrm{p}}L} \equiv \left| e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(\Omega)} \right|^2, \tag{17}$$

we can now rewrite Eq. (14) as

$$\frac{\tilde{\mathcal{E}}_{\mathrm{s}}^{\mathrm{out}}(\Omega)}{\tilde{\mathcal{E}}_{\mathrm{s}}^{\mathrm{in}}(\Omega)e^{-i\Omega T}} = e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(\Omega)}. \tag{18}$$

It is not possible to perform rigorously an inverse Fourier transform for the fields $\tilde{\mathcal{E}}_{\mathrm{s}}^{\mathrm{in,out}}(\Omega)$ in Eq. (18). The main reason is that there is no closed-form relationship giving the inverse Fourier transform for the exponential of Lorentzian. However, one can note that for small gain (i. e. $g_{\mathrm{B0}}I_{\mathrm{p}}L \ll 1$), the exponential of a Lorentzian can be approximated to a Lorentzian superimposed to a flat background, following

$$e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(\Omega)} \simeq 1 + \frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(\Omega) \tag{19}$$

which allows to perform a closed-form inverse Fourier transform for the field $\tilde{\mathcal{E}}_{\mathrm{s}}^{\mathrm{in,out}}(\Omega)$ in Eq. (18).

Because of this useful mathematical property (the possibility to perform an inverse Fourier transform), we propose to generalize the small-gain expansion of Eq. (19) to the case of large gain. We therefore consider that gain is the sum of a flat background (= 1) and a Lorentzian, even in the case of large gain, that is

$$e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(\Omega)} \simeq 1 + G_{\mathrm{o}}\mathcal{L}_{\mathrm{eq}}(\Omega) \quad \text{with} \quad \mathcal{L}_{\mathrm{eq}}(\Omega) = \frac{\mu}{\mu + i\Omega}, \tag{20}$$

where $\mathcal{L}_{\mathrm{eq}}(\Omega)$ is an equivalent Lorentzian with full width half maximum bandwidth 2μ, while G_{o} is the equivalent peak gain. The rationale behind the approximation of Eq. (20) is twofold. The first reason is that as explained earlier, we can find a closed-form analytical formula for the inverse Fourier transform for a Lorentzian gain, while we cannot do so for the exponential of a Lorentzian. The second reason is related to the experimental features if Brillouin grain: Indeed, in practice, the gain profile $g_{\mathrm{B}}(\Omega)$ of the Brillouin gain is only *approximated* by a Lorentzian, so that strictly speaking, expressing our gain as the exponential of the Brillouin Lorentzian in Eq. (14) is also an approximation, and *not* an exact result; therefore, the choice of the fitting function for the Brillouin amplification is not critical as long as that function displays the main features of frequency-selective amplification (input signal is unaltered outside of bandwidth, while it is amplified when within bandwidth; see for example [43,44]).

For the approximation of Eq. (20) to be physically valid, we have to make sure that the peak value and half-linewidth values of the equivalent Lorentzian gain $1 + G_{\mathrm{o}}\mathcal{L}_{\mathrm{eq}}(\Omega)$ match those of the "exact" profile $e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(\Omega)}$. We therefore have the equalities

$$\textit{Gain peak condition:} \quad \left| e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(0)} \right| = \left| 1 + G_{\mathrm{o}}\mathcal{L}_{\mathrm{eq}}(0) \right| \tag{21}$$

$$\textit{Gain bandwidth condition:} \quad \left| e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L \, \mathcal{L}_{\mathrm{B}}(\pm\mu)} \right| = \left| 1 + G_{\mathrm{o}}\mathcal{L}_{\mathrm{eq}}(\pm\mu) \right| \tag{22}$$

that lead to

$$G_{\mathrm{o}} = e^{\frac{1}{2} g_{\mathrm{B0}}I_{\mathrm{p}}L} - 1 \tag{23}$$

$$\mu = \frac{\Delta\Omega_B}{2}\sqrt{\frac{g_{B0}I_p L}{\ln\left[\frac{1}{2}\left(e^{g_{B0}I_p L}+1\right)\right]}-1}\,. \tag{24}$$

Note that unlike Brillouin scattering that has a peak gain and bandwidth independent of the pump intensity I_p, frequency-selective Brillouin amplification has a pump dependent peak gain and bandwidth.

3.4. Time-domain equations for Brillouin amplification

We now have all the elements needed to perform the inverse Fourier transform that will give the us the equations ruling the field dynamics in the time domain. Let us first rewrite the output signal as

$$\mathcal{E}_s^{out}(t) = \mathcal{E}_s^{amp}(t) + \mathcal{E}_s^{in}(t-T) \tag{25}$$

where $\mathcal{E}_s^{amp}(t)$ is the amplified portion of the output signal. Then, we can use Eqs. (18) and (20) to express the Fourier transform of this amplified portion as

$$\begin{aligned}
\tilde{\mathcal{E}}_s^{amp}(\Omega) &= \tilde{\mathcal{E}}_s^{out}(\Omega) - \tilde{\mathcal{E}}_s^{in}(\Omega)e^{-i\Omega T} \\
&= G_o \frac{\mu}{\mu+i\Omega}\,\tilde{\mathcal{E}}_s^{in}(\Omega)e^{-i\Omega T}\,.
\end{aligned} \tag{26}$$

After multiplying both sides of this equation by $\mu + i\Omega$, we can perform an inverse Fourier transform and finally obtain the desired equation

$$\dot{\mathcal{E}}_s^{amp}(t) = -\mu\mathcal{E}_s^{amp}(t) + \mu G_o\,\mathcal{E}_s^{in}(t-T) \tag{27}$$

where the overdot stands for time derivative. Equation (27) rules the frequency-selective amplification in the time domain, and the output signal $\mathcal{E}_s^{out}(t)$ is recovered via Eq. (25). It is interesting to note that we have mapped the partial differential Eq. (9) into a time-delayed equation, while adding the narrowband filtering property of the amplifier.

It is interesting to note that: *(i)* We do not need to add a detuning σ; *(ii)* We adopt an input-output convention such that the excitation term is not multiplied by i; *(iii)* By definition, the initial condition for $\mathcal{E}_s^{amp}$ is null, i.e. $\mathcal{E}_s^{amp}(0) = 0$.

4. OEO model

4.1. Phase modulation

The signal laser emits a monochromatic wave, that can be written as

$$\mathcal{E}_L = \sqrt{P_0}\,, \tag{28}$$

where P_0 is the optical power of this continuous-wave laser (i.e., $\mathcal{E}_L$ is in units of $\sqrt{W}$). We assume that this phase modulator has an RF modulation signal $V(t)$, so that its output optical field is

$$\mathcal{E}_{PM}(t) = \mathcal{E}_L \exp\left\{i\pi\frac{V(t)}{V_{\pi_p}}\right\} \tag{29}$$

$$= \sqrt{P_0}\exp\left\{i\pi\frac{|\mathcal{V}(t)|}{V_{\pi_p}}\cos[\Omega_0 t + \psi(t)]\right\}\,. \tag{30}$$

The Jacobi-Anger expansion gives

$$e^{ix\cos\alpha} = \sum_{n=-\infty}^{+\infty} i^n J_n(x)e^{in\alpha} \tag{31}$$

where J_n is the *n*-th order Bessel function of the first kind. Therefore, we have the expansion

$$\mathcal{E}_{\mathrm{PM}}(t) = \sqrt{P_0}\,\mathcal{E}(t) = \sqrt{P_0}\sum_{n=-\infty}^{+\infty}\mathcal{E}_n(t)e^{in\Omega_0 t} \tag{32}$$

with the modal fields

$$\mathcal{E}_n(t) = i^n \mathrm{J}_n\left[\pi\frac{|\mathcal{V}(t)|}{V_{\pi_p}}\right]e^{in\psi(t)}. \tag{33}$$

Note that the complex envelope field $\mathcal{E}$ and its modal components $\mathcal{E}_n$ are *dimensionless*.

The first sidemode $\mathcal{E}_1(t)$ is now injected into the Brillouin amplifier. Following the analysis led in the preceding Section, the amplified portion of the signal in the fiber obeys the equation

$$\dot{\mathcal{B}}(t) = -\mu\mathcal{B}(t) + \mu G_o \mathcal{E}_1(t-T) \tag{34}$$

with initial condition $\mathcal{B}(0) = 0$ [also note that $\mathcal{B}(t)$ is a dimensionless variable]. The total output optical field after Brillouin amplification is now

$$\mathcal{F}(t) = \sum_{n=-\infty}^{+\infty}\mathcal{F}_n(t)e^{in\Omega_0 t}, \tag{35}$$

with $\mathcal{F}_n(t)$ being defined as

$$\mathcal{F}_n(t) = \mathcal{E}_n(t-T) + \delta(n-1)\mathcal{B}(t) \equiv \begin{cases} \mathcal{E}_1(t-T) + \mathcal{B}(t) & \text{for } n = 1 \\ \mathcal{E}_n(t-T) & \text{otherwise} \end{cases}, \tag{36}$$

where $\delta(x)$ is the Kronecker function (equal to 1 for $x = 0$ and to 0 otherwise), while all the modal fields $\mathcal{E}_n(t)$ are defined as in Eq. (33).

4.2. Accounting for the Brillouin amplification

Now that we have the equation ruling the Brillouin amplification, we can proceed with calculating the voltage generated by the photodetection of this optical signal, which can be expressed as (in units of V):

$$V_{\mathrm{pd}}(t) = SP_0|\mathcal{F}(t)|^2 = SP_0\left|\sum_{n=-\infty}^{+\infty}\mathcal{F}_n(t)e^{in\Omega_0 t}\right|^2$$

$$= \frac{1}{2}\mathcal{M}_0(t) + \sum_{k=1}^{+\infty}\left\{\frac{1}{2}\mathcal{M}_k(t)e^{ik\Omega_0 t} + \text{c.c.}\right\},$$

where c. c. stands for the complex conjugate of the preceding terms, and

$$\mathcal{M}_n(t) = 2SP_0\sum_m \mathcal{F}_m^*(t)\mathcal{F}_{m+n}(t) \tag{37}$$

is the complex slowly-varying envelope corresponding to the microwave spectral component of frequency $n \times \Omega_0$ (in volts), while S stands for the photodiode sensitivity (in units of V/W). This photodetected signal is then bandpass filtered, so that it rejects the DC component $\mathcal{M}_0$ and the harmonics $\mathcal{M}_k$ with $k \geq 2$. Hence, only the spectral component at frequency Ω_0 is allowed to pass through (note that this filter is *not* narrow, so that it does not specifically induce dynamical effects other than fundamental mode selection). The slowly-varying amplitude of the voltage at the output of the photodiode is therefore $\mathcal{M}_1(t)$ (still in units of V). This signal is then amplified in the electrical branch with a gain G_e (dimensionless). We can also account for all the loop

losses through a single loss coefficient κ (dimensionless). Therefore, the slowly-varying envelope $\mathcal{V}(t)$ of the microwave signal fed back to the RF input of the phase modulator is

$$
\begin{aligned}
\mathcal{V}(t) &= \kappa G_{\mathrm{e}} e^{i\Phi} \mathcal{M}_1(t) \\
&= 2\kappa G_{\mathrm{e}} S P_0 e^{i\Phi} \sum_{n=-\infty}^{+\infty} \mathcal{F}_n^*(t) \mathcal{F}_{n+1}(t),
\end{aligned}
\tag{38}
$$

where the phase factor $e^{i\Phi}$ accounts for the effect of the microwave round-trip phase shift Φ. If necessary, this phase be tuned to any desired value (modulo 2π) using an RF phase shifter.

We now need to put all these elements together in order to obtain our final model. Let us introduce the dimensionless microwave envelope

$$
\mathcal{A}(t) = \frac{\pi}{V_{\pi_p}} \mathcal{V}(t)
\tag{39}
$$

and the optoelectronic gain as

$$
\beta = \frac{\pi \kappa G_{\mathrm{e}} S P_0}{V_{\pi_p}}.
\tag{40}
$$

Note that β is *not* a loop-gain parameter as it is the case for conventional fiber-based OEOs, since it explicitly excludes the Brillouin amplification.

We therefore have the following four-step model for our tunable OEO based on Brillouin amplification:

$$
\mathcal{E}_n(t) = i^n \, \mathrm{Jc}_n[|\mathcal{A}(t)|] \mathcal{A}^n(t)
\tag{41}
$$

$$
\dot{\mathcal{B}}(t) = -\mu \mathcal{B}(t) + \mu G_{\mathrm{o}} \mathcal{E}_1(t-T)
\tag{42}
$$

$$
\mathcal{F}_n(t) = \mathcal{E}_n(t-T) + \delta(n-1)\mathcal{B}(t)
\tag{43}
$$

$$
\mathcal{A}(t) = 2\beta e^{i\Phi} \sum_{n=-\infty}^{+\infty} \mathcal{F}_n^*(t) \mathcal{F}_{n+1}(t)
\tag{44}
$$

where

$$
\mathrm{Jc}_n(x) = \frac{\mathrm{J}_n(x)}{x^n} \quad \text{with } x \in \mathbb{R} \text{ and } n \in \mathbb{Z},
\tag{45}
$$

is the Bessel-cardinal function of order n.

Note that the gain of this OEO can be tuned in this system following two different ways: either via the optical gain G_{o} (controlled by the power of the pump laser), or via the optoelectronic gain β (controlled via the gain G_{e} of the RF amplifier of the electronic branch). In this study, we will consider without loss of generality that the optical gain G_{o} is fixed, while the optoelectronic gain β is tunable via G_{e}.

4.3.　Envelope equation for the generic case $T \neq 0$

The four-step model presented in the preceding subsection can be simplified. We have indeed

$$
\mathcal{A}(t) = 2\beta e^{i\Phi} \sum_{n=-\infty}^{+\infty} \mathcal{F}_n^*(t) \mathcal{F}_{n+1}(t)
\tag{46}
$$

$$
= 2\beta e^{i\Phi} \left\{ \mathcal{B}(t) \mathrm{J}_0[|\mathcal{A}(t-T)|] - \mathcal{B}^*(t) \mathrm{Jc}_2[|\mathcal{A}(t-T)|] \mathcal{A}^2(t-T) \right\}
\tag{47}
$$

$$
+ 2i\beta e^{i\Phi} e^{i\psi(t-T)} \sum_{n=-\infty}^{+\infty} \mathrm{J}_n[|\mathcal{A}(t-T)|] \mathrm{J}_{n+1}[|\mathcal{A}(t-T)|],
\tag{48}
$$

but however, Bessel functions of the first kind obey

$$\sum_{n=-\infty}^{+\infty} J_n(x)J_{n+m}(x) \equiv 0 \tag{49}$$

for all $x \in \mathbb{R}$ and all $m \in \mathbb{Z}$. Knowing that $\mathcal{E}_1 = i\,\mathrm{Jc}_1[|\mathcal{A}|]\mathcal{A}$, we can now rewrite the four-step model under the simpler form

$$\dot{\mathcal{B}} = -\mu\mathcal{B} + i\mu G_\mathrm{o}\mathrm{Jc}_1[|\mathcal{A}_T|]\mathcal{A}_T \tag{50}$$

$$\mathcal{A} = 2\beta e^{i\Phi}\Big\{ \mathcal{B}J_0[|\mathcal{A}_T|] - \mathcal{B}^*\mathrm{Jc}_2[|\mathcal{A}_T|]\mathcal{A}_T^2 \Big\}. \tag{51}$$

Equation (51) can be furthermore simplified. Indeed, an analysis of the above equations shows that the phases ψ and φ of $\mathcal{A}$ and $\mathcal{B}$ are such that

$$\varphi(t) = \frac{\pi}{2} + \psi(t-T) \quad \text{and} \quad \psi(t) = \Phi + \varphi(t) \tag{52}$$

which leads to the solution

$$\Phi = -\frac{\pi}{2} \quad \text{and} \quad \psi(t) = \varphi(t) - \frac{\pi}{2} = \text{Constant}. \tag{53}$$

Note that a direct consequence of this solution is that we now have $e^{i\Phi} = -i$, so that a phase shifter has to be introduced in the loop to implement this phase condition. A phase rotation therefore allows to obtain

$$\mathcal{A} = -2i\beta\mathcal{B}\Big\{ J_0[|\mathcal{A}_T|] + J_2[|\mathcal{A}_T|] \Big\} = -4i\beta\mathcal{B}\,\mathrm{Jc}_1[|\mathcal{A}_T|], \tag{54}$$

where we have used the recurrence relationship

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2nJ_n(x)}{x}. \tag{55}$$

Note that Eq. (54) preserves the phase relationship of Eq. (53).

Let us now rewrite Eqs. (54) and (50) as

$$\mathcal{B} = -\frac{\mathcal{A}}{4i\beta\,\mathrm{Jc}_1[|\mathcal{A}_T|]} \tag{56}$$

$$\dot{\mathcal{B}} = \frac{\mu\mathcal{A}}{4i\beta\,\mathrm{Jc}_1[|\mathcal{A}_T|]} + i\mu G_\mathrm{o}\,\mathrm{Jc}_1[|\mathcal{A}_T|]\mathcal{A}_T \tag{57}$$

meaning that both $\mathcal{B}$ and $\dot{\mathcal{B}}$ are expressed completely as a function of $\mathcal{A}$ and $\mathcal{A}_T$.

We can now determine $\dot{\mathcal{A}}$ as

$$\dot{\mathcal{A}} = \frac{\partial\mathcal{A}}{\partial\mathcal{B}}\frac{\partial\mathcal{B}}{\partial t} + \frac{\partial\mathcal{A}}{\partial|\mathcal{A}_T|}\frac{\partial|\mathcal{A}_T|}{\partial t} \tag{58}$$

$$= \frac{\partial\mathcal{A}}{\partial\mathcal{B}}\dot{\mathcal{B}} + \frac{\partial\mathcal{A}}{\partial|\mathcal{A}_T|}\partial_t|\mathcal{A}_T|, \tag{59}$$

where

$$\partial_t|\mathcal{A}_T| = \partial_t[\mathcal{A}_T\mathcal{A}_T^*]^{\frac{1}{2}} = \frac{\dot{\mathcal{A}}_T\mathcal{A}_T^* + \mathcal{A}_T\dot{\mathcal{A}}_T^*}{2|\mathcal{A}_T|} \tag{60}$$

is the derivative of $|\mathcal{A}_T|$ with regard to time. It is important to note that $\partial_t|\mathcal{A}_T| \neq |\partial_t\mathcal{A}_T| = [\dot{\mathcal{A}}_T\dot{\mathcal{A}}_T^*]^{\frac{1}{2}}$. From Eq. (54), we obtain

$$\frac{\partial\mathcal{A}}{\partial\mathcal{B}} = -4i\beta\,\mathrm{Jc}_1[|\mathcal{A}_T|] \tag{61}$$

$$\frac{\partial\mathcal{A}}{\partial|\mathcal{A}_T|} = -4i\beta\mathcal{B}\mathrm{Jc}_1'[|\mathcal{A}_T|]. \tag{62}$$

We now use the results from Eqs. (57), (56), (61), and (62) to rewrite Eq. (59) as

$$\dot{\mathcal{A}} = -4i\beta\,\mathrm{Jc}_1[|\mathcal{A}_T|]\left\{\frac{\mu\mathcal{A}}{4i\beta\,\mathrm{Jc}_1[|\mathcal{A}_T|]} + i\mu G_o\,\mathrm{Jc}_1[|\mathcal{A}_T|]\mathcal{A}_T\right\}$$
$$- 4i\beta\mathrm{Jc}_1'[|\mathcal{A}_T|]\,\partial_t|\mathcal{A}_T|\left\{-\frac{\mathcal{A}}{4i\beta\,\mathrm{Jc}_1[|\mathcal{A}_T|]}\right\}, \tag{63}$$

and using $\mathrm{Jc}_1'(x) = -\mathrm{J}_2(x)/x$, this can be finally be written as

$$\dot{\mathcal{A}} = -\mu\mathcal{A} + 4\mu\Gamma\,\mathrm{Jc}_1^2[|\mathcal{A}_T|]\mathcal{A}_T - \left\{\frac{\mathrm{J}_2[|\mathcal{A}_T|]}{\mathrm{J}_1[|\mathcal{A}_T|]}\,\partial_t|\mathcal{A}_T|\right\}\mathcal{A} \tag{64}$$

with $\Gamma = \beta G_o$ being the loop gain.

Equation (64) is well-defined, and in particular, the time-delayed terms in the brackets in the right-hand side are well-defined at any time. The term $\partial_t|\mathcal{A}_T|$ is indeed unusual, but is not problematic from the mathematical viewpoint because at any time t, the variable $\mathcal{A}_T$ and any variable that univocally depends on it can be determined unambiguously, even at the numerical level. One can also note that in the steady state, we have $\dot{\mathcal{A}} = 0$, $\partial_t|\mathcal{A}_T| = 0$. Note that unlike the conventional envelope equations for OEOs, Eq. (64) involves coupling between the delayed and non-delayed variables. Also note that the phase dynamics is irrelevant in the deterministic case, and can be disregarded but will become key at the time to investigate the noise performance of the system [45–49].

4.4. Envelope equation for the particular case $T = 0$

Using Eq. (64), we can obtain an envelope equation for the case $T = 0$, which corresponds to the physical configuration where the system is chipscale (see Fig. 2). We first note that the phase ψ of the microwave envelope $\mathcal{A}$ is a constant of motion [see Eq. (53)]. It results that $[\partial_t|\mathcal{A}|]\,\mathcal{A} = \dot{\mathcal{A}}|\mathcal{A}|$ and consequently, Eq. (64) can be written as

$$\dot{\mathcal{A}} = -\mu\mathcal{A} + 4\mu\Gamma\,\mathrm{Jc}_1^2[|\mathcal{A}|]\mathcal{A} - \left\{\frac{\mathrm{J}_2[|\mathcal{A}|]}{\mathrm{J}_1[|\mathcal{A}|]}\,|\mathcal{A}|\right\}\dot{\mathcal{A}}, \tag{65}$$

from which we directly obtain

$$\dot{\mathcal{A}} = -\mu\mathcal{A}\,\frac{1 - 4\Gamma\,\mathrm{Jc}_1^2[|\mathcal{A}|]}{1 + \mathrm{J}_2[|\mathcal{A}|]/\mathrm{Jc}_1[|\mathcal{A}|]}. \tag{66}$$

This equation is the one that has to be considered to track the temporal dynamics of the microwave envelope in the case where $T = 0$.

5. Stationary states and their stability for $T = 0$

We already know that the phase of $\mathcal{A}$ is an arbitrary constant, which can be set without loss of generality to zero. In that case, $\mathcal{A}$ becomes real-valued and we can remove the calligraphic fonts following $\mathcal{A} \equiv A \geq 0$, so that the amplitude equation reads

$$\dot{A} = -\mu A \frac{1 - 4\Gamma \, \text{Jc}_1^2[A]}{1 + \text{J}_2[A]/\text{Jc}_1[A]} \, . \tag{67}$$

The stationary state of Eq. (67) can be obtained as

$$\frac{A_{\text{st}}}{4\Gamma} = \frac{\text{J}_1^2[A_{\text{st}}]}{A_{\text{st}}} \tag{68}$$

and two possible solutions are

$$A_{\text{st}} = \begin{cases} A_{\text{tr}} = 0 \\ A_{\text{osc}} = \text{Jc}_1^{-1} \left[\pm \frac{1}{2\sqrt{\Gamma}} \right] \end{cases} \tag{69}$$

where $A_{\text{tr}} = 0$ is the trivial solution, $A_{\text{osc}} \neq 0$ is the oscillatory (nontrivial) solution, and Jc_1^{-1} is the inverse function of Jc_1 (whenever single-valued). The above solutions are valid regardless of there is a time-delay or not. Equation (68) and (69) are graphically shown in Fig. 3. The trivial solution exists for the all gains while the oscillatory solution is valid for $\Gamma > 1$.

Fig. 3. The graphical representation of Eq. (68) where $L(A_{\text{st}}) = A_{\text{st}}/4\Gamma$ (linear in A_{st}) and $NL(A_{\text{st}}) = \text{J}_1^2[A_{\text{st}}]/A_{\text{st}}$ (nonlinear in A_{st}) are expressed as discontinuous colored and continuous black lines respectively.

Equation (69) shows two possible oscillatory solutions, but we can reduce to one by confirming the limit of the Bessel-Cardinal function. The function oscillates between $-0.066 < \text{Jc}_1(x) < 0.5$. If we consider Γ is a positive value, the two solutions should cover the positive and negative range separately, and thus

$$\text{Jc}_1[A_{\text{osc}}] = \begin{cases} +\frac{1}{2\sqrt{\Gamma}} & \text{for} & 0 < \text{Jc}_1[A_{\text{osc}}] < 0.5 \\ -\frac{1}{2\sqrt{\Gamma}} & \text{for} & -0.066 < \text{Jc}_1[A_{\text{osc}}] < 0 \end{cases} \tag{70}$$

The only requirement for the validity of the positive solution is $\Gamma > 1$ while the negative solution needs $\Gamma > 56.0$ practically very difficult to achieve. We can only consider the positive solution,

and therefore, the oscillatory solution A_{osc} we will consider is

$$A_{osc} = \text{Jc}_1^{-1} \left[\frac{1}{2\sqrt{\Gamma}} \right] \quad \text{where} \quad 1 < \Gamma < 56.0. \tag{71}$$

Now, we need to determine the stability of the two solutions. However, since the upper limit $\Gamma \simeq 56$ is already far beyond physically achievable, we are going to restrict our analysis to the range $0 \leq \Gamma < 56$ for the gain values.

5.1. Stability of the trivial solution

To evaluate the stability of the trivial solution $A_{tr} = 0$, we can perturb the fixed point and check if it decays or grows by depending on the loop gain Γ. Considering a very small perturbation δA around the fixed point leads Eq. (67) to be written as

$$\delta\dot{A} = -\mu(1 - \Gamma)\,\delta A \tag{72}$$

where higher order nonlinear terms are neglected. It is trivial to show that δA decays to zero for $\Gamma < 1$, while it diverges for $\Gamma < 1$. Therefore, feedback gain $\Gamma_{th} \equiv 1$ defines the threshold below which the trivial fixed point $A_{tr} = 0$ remains stable (no oscillations), and above which microwave oscillations are triggered (microwave oscillation of frequency Ω_0 and constant amplitude). It can be shown that this bifurcation corresponds to a Hopf bifurcation for the microwave signal, and a pitchfork bifurcation for its amplitude [1,17–19].

5.2. Stability of the oscillating solution

We now perturb the oscillating solution A_{osc} (which only exists for $\Gamma > 1$), so that the linearized counterpart of Eq. (67) can be written as

$$\delta\dot{A} = -\mu\Gamma R_1\,\delta A \quad \text{with.} \quad R_1 = \frac{8\text{Jc}_1[A_{osc}]\text{J}_2[A_{osc}]}{1 + \text{J}_2[A_{osc}]/\text{Jc}_1[A_{osc}]}. \tag{73}$$

Since the constant coefficient R_1 is always positive, it appears that we always have $\delta A \rightarrow 0$ for $1 < \Gamma < 56$, so that the oscillatory solution A_{osc} is always stable for that range of gain values.

5.3. Numerical simulation for the case $T = 0$

Figures 4(a) and (b) display the transient dynamics of the SBS-based OEO with null delay for $\Gamma = 0.9$ and $\Gamma = 1.1$, that is, just below and above the threshold value $\Gamma_{th} = 1$. On the other hand, Figs. 4(c) and (d) show the dynamics of the oscillator at $\Gamma = 2.49$ and $\Gamma = 56$, thereby showing that oscillating solution is indeed stable in the gain range of interest in for that solution ($1 < \Gamma < 56$). The amplitudes of the oscillating solutions also correspond to the intersections points in Fig. 3.

Fig. 4. Numerical simulation of the SBS-based OEO with null delay [Eq. (66)] for various values of the gain Γ. The value of the Brillouin gain bandwidth is set to $\mu = 2\pi \times 25$ MHz.

6. Stationary states and their stability for $T \neq 0$

We now consider the generic case where the delay is not null. Here also we can take advantage of the fact that that the phase ψ of the microwave is a constant. Once again it is convenient to arbitrarily set $\psi = 0$, so that the envelope $\mathcal{A}$ becomes real-valued, with $\mathcal{A} \equiv A > 0$ and $\mathcal{A}_T \equiv A_T > 0$. The microwave dynamics is now ruled by

$$\dot{A} = -\mu A + 4\mu\Gamma \, \mathrm{Jc}_1^2[A_T]A_T - \frac{\mathrm{J}_2[A_T]}{\mathrm{J}_1[A_T]} A\dot{A}_T \,, \tag{74}$$

It first appears that the stationary states of Eq. (74) are exactly the same as those obtained for the case $T = 0$. We have trivial and oscillatory solutions given by Eq. (69), and the restriction discussed in Eq. (71) also apply in the present case.

We can determine the stability of the two solutions in the same way as $T = 0$ case.

6.1. Stability of the trivial solution

When we perturb the trivial solution with $\delta A = \delta A_0 \exp[(\lambda_{\mathrm{tr}} + i\xi_{\mathrm{tr}})t]$, the coefficients λ_{tr} and the ξ_{tr} will obey

$$\lambda_{\mathrm{tr}} = -\mu + \mu\Gamma e^{-\lambda_{\mathrm{tr}}T} \cos \xi_{\mathrm{tr}} T \tag{75}$$

$$\xi_{\mathrm{tr}} = -\mu\Gamma e^{-\lambda_{\mathrm{tr}}T} \sin \xi_{\mathrm{tr}} T. \tag{76}$$

We can only consider near $\lambda_{\mathrm{tr}} = 0$ where the sign change of the λ_{tr} occurs. Equation (76) generates two possible solutions $\xi_{\mathrm{tr}} = 0$ and $\xi_{\mathrm{tr}} = \pi/T$ but the later one cannot satisfy Eq. (75). Therefore, the unique solution is

$$\lambda_{\mathrm{tr}} \simeq \frac{\Gamma - 1}{T\Gamma} \tag{77}$$

$$\xi_{\mathrm{tr}} = 0 \tag{78}$$

where we have assumed the condition $\mu T \gg 1$. This solution generates the same result as for the case $T = 0$, i.e. the trivial solution is stable for $\Gamma < \Gamma_{\mathrm{th}}$ and unstable otherwise.

6.2. Stability of the oscillating solution

When we perturb the oscillatory solution A_{osc}, the linearization of Eq. (74) yields

$$\delta\dot{A} = -\mu\delta A + \mu R_2 \, \delta A_T - R_3 \, \delta\dot{A}_T \tag{79}$$

where

$$R_2 = 8\Gamma \text{Jc}_1[A_{\text{osc}}]\text{J}_0[A_{\text{osc}}] - 3 \quad \text{and} \quad R_3 = \text{J}_2[A_{\text{osc}}]/\text{Jc}_1[A_{\text{osc}}]. \tag{80}$$

If we define $\delta A = \delta A_0 \exp[(\lambda_{\text{osc}} + i\xi_{\text{osc}})t]$, the real and imaginary parts will be ruled by

$$\lambda_{\text{osc}} = \frac{-\mu + \mu R_2 e^{-\lambda_{\text{osc}}T}\cos\xi_{\text{osc}}T - R_3 e^{-\lambda_{\text{osc}}T}\xi_{\text{osc}}\sin\xi_{\text{osc}}T}{1 + R_3 e^{-\lambda_{\text{osc}}T}\cos\xi_{\text{osc}}T} \tag{81}$$

$$\xi_{\text{osc}} = \frac{-(\mu R_2 - R_3\lambda_{\text{osc}})e^{-\lambda_{\text{osc}}T}\sin\xi_{\text{osc}}T}{1 + R_3 e^{-\lambda_{\text{osc}}T}\cos\xi_{\text{osc}}T} \tag{82}$$

and it leads to two possible solutions

$$\lambda_{\text{osc}} \simeq \begin{cases} \frac{R_2-1}{R_2 T} & \text{if } R_2 > 0 \\ \frac{R_2+1}{R_2 T} & \text{if } R_2 < 0 \end{cases} \tag{83}$$

where we are still assuming $\mu T \gg 1$. Equation (83) shows that λ_{osc} is negative for $|R_2| < 1$, and that condition corresponds to the gain range $1 < \Gamma < 2.49$. Therefore, if we define $\Gamma_{\text{cr}} \equiv 2.49$ as the critical value for which the microwave is destabilized, then the oscillatory solution is stable for $\Gamma_{\text{th}} < \Gamma < \Gamma_{\text{cr}}$.

The secondary bifurcation occurring at $\Gamma_{\text{cr}} = 2.49$ is a Neimark-Sacker or *torus* bifurcation [1,17,19,50,51]. It should be noted that in the case of a short (but non-null) delay with $\mu T \sim 1$, the formalism developed above is not valid anymore and a more involved calculation would be needed to determine Γ_{cr} – see Ref. [52].

6.3. Numerical simulation for the case $T \neq 0$

Figures 5(a) and (b) display the transient dynamics of $\Gamma = 0.9$ and $\Gamma = 1.1$ the just below and above the threshold value Γ_{th}. On the other hand, Figs. 5(c) and (d) show the dynamics of the oscillator at $\Gamma = 2.48$ and $\Gamma = 2.5$, that is, just before and just after the torus bifurcation. The amplitude of the output microwave is stable below the critical value (Γ_{cr}) and unstable above, as predicted by the bifurcation analysis. When the gain is further increased, the oscillator displays $4T$-periodic behavior [Fig. 5(e)] while further increase of Γ leads to a chaotic behavior. [Fig. 5(f)]. The overall dynamics of the oscillator can be described via a bifurcation diagram as displayed in Fig. 6, where the Hopf and Neimark-Sacker bifurcations are shown to occur exactly where they where analytically predicted.

Fig. 5. Numerical simulation of the SBS-based OEO when $T \neq 0$ [Eq. (74)] for various values of the gain Γ. The insets of (c) and (d) show the asymptotic dynamical behavior of the system, evidencing the qualitative difference induced by the Neimark-Sacker bifurcation at $\Gamma_{\text{cr}} = 2.49$. The parameters of the system are $\mu = 2\pi \times 25$ MHz and $T = 5$ μs.

Fig. 6. Bifurcation diagram for the SBS-based OEO. The labels (a)–(f) indicate the values of the gain Γ used for the time-domain simulations in Fig. 5. The Hopf and Neimark-Sacker bifurcations numerically emerge at the gain values $\Gamma_{\text{th}} = 1$ and $\Gamma_{\text{cr}} = 2.49$ (resp.), as predicted by the theory. The parameters of the system are $\mu = 2\pi \times 25$ MHz and $T = 5$ μs.

7. Conclusion

In this article, we have studied the nonlinear dynamics of SBS-based OEOs. We have first analyzed the SBS effect in the nonlinear fiber acting as a microwave photonic filter, and we have derived as well the differential equation ruling the input-output relationship of frequency-selective amplification. This modelling of the SBS process has enabled us to obtain a delay-differential equation governing the microwave envelope dynamics of the OEO output signal. We have then performed a stationary state analysis in order to find the various solutions (trivial and oscillatory), and to evaluate their stability. This procedure has led us to the analytical determination of the threshold gain value triggering the onset of microwave oscillations, as well as the critical value leading to their destabilization via amplitude modulation. Our numerical simulations have confirmed the stability analysis.

Future research will specifically focus on the phase noise optimization of these oscillations. This endeavor will include a detailed analysis of the beneficial effects of Brillouin scattering with regard to the close-in phase noise performance of the output microwave.

Disclosures. The authors declare no conflicts of interest.

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